

# From Thermodynamics to the Bound on Viscosity

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We show that the generalized second law of thermodynamics may shed much light on the mysterious Kovtun-Son-Starinets (KSS) bound on the ratio of viscosity to entropy density. In particular, we obtain the lower bound  $\eta/s + O(\eta^3/s^3) \geq 1/4\pi$ . Furthermore, for conformal field theories we obtain a new fundamental bound on the value of the relaxation coefficient  $\tau_\pi$  of causal hydrodynamics, which has been the focus of much recent attention:  $(\tau_\pi T)^2 \geq (\sqrt{3} - 1)/2\pi^2$ .

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1, 2, 3, 4] has yielded remarkable insights into the dynamics of strongly coupled gauge theories. According to this duality, asymptotically AdS background spacetimes with event horizons are interpreted as thermal states in dual field theories. This implies that small perturbations of a black hole or a black brane background correspond to small deviations from thermodynamic equilibrium in a dual field theory. One robust prediction of the AdS/CFT duality is a universally small ratio of the shear viscosity to the entropy density [5, 6, 7, 8],

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad (1)$$

for all gauge theories with an Einstein gravity dual in the limit of large 't Hooft coupling. (We use natural units for which  $G = c = \hbar = k_B = 1$ .)

It was suggested [8] that (1) acts as a universal lower bound [the celebrated Kovtun-Starinets-Son (KSS) bound] on the ratio of the shear viscosity to the entropy density of general, possibly nonrelativistic, fluids. Currently this bound is considered a *conjecture* well supported for a certain class of field theories—see the detailed discussions in [9, 10] and the references therein. So far, all known materials satisfy the bound for the range of temperatures and pressures examined in the laboratory. The system coming closest to the bound is the quark-gluon plasma created at the BNL Relativistic Heavy Ion Collider (RHIC) [11, 12, 13, 14]. [In fact, it was the challenge presented by the quark-gluon plasma which motivated the activity leading to the formulation of the KSS bound (1).] Other systems coming close to the bound include superfluid helium and trapped  ${}^6\text{Li}$  at strong coupling [15, 16]. For other related works, see [17, 18, 19, 20, 21] and references therein.

It is important to note that recent work [22] has shown that, for a class of conformal field theories with Gauss-Bonnet gravity dual, the shear viscosity to entropy density ratio,  $\eta/s$ , could violate the conjectured KSS bound. In particular, for (3+1)-dimensional CFT duals of (4+1)-dimensional Gauss-Bonnet gravity, the ratio  $\eta/s$  is given

by

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 - 4\lambda_{GB}), \quad (2)$$

where  $\lambda_{GB}$  is the Gauss-Bonnet coupling parameter [23]. It was later shown that consistency of the theory requires  $\lambda_{GB} \leq \frac{9}{100}$  [4]. This still leaves room for a violation of the KSS bound (1). This observation suggests that, if there is indeed a *universal* lower bound on the ratio  $\eta/s$ , then it is likely to be a bit more liberal than the originally conjectured KSS bound. This is exactly the kind of result we shall find below. It should be noted, however, that there is no known quantum field theory whose hydrodynamic regime coincides with particularly chosen gravitational lagrangian. So it might also be the case that the KSS bound is robust and the Gauss-Bonnet lagrangian does not capture the hydrodynamic limit of a field theory.

Where does the KSS bound (or any other refined bound on the ratio  $\eta/s$ ) come from? It is not clear how to obtain such a bound directly from microscopic physics [10]. Inspection of the Green-Kubo formula [24] which relates the viscosity of a fluid to its fluctuations shows no apparent connection of the viscosity and the entropy density. Such microscopic consideration affords no special status to the ratio  $\eta/s$  [10].

Where should we look for the physical mechanism which bounds the ratio  $\eta/s$  of viscosity to entropy density? It is well known that the viscosity coefficient  $\eta$  characterizes the intrinsic ability of a perturbed fluid to relax towards equilibrium [25] [see Eqs. (3) and (9) below]. The response of a medium to mechanical excitations is characterized by two types of normal modes, corresponding to whether the momentum density fluctuations are transverse or longitudinal to the fluid flow. Transverse fluctuations lead to the shear mode, whereas longitudinal momentum fluctuations lead to the sound mode. (There is also the diffuse mode in the presence of a conserved current.) These perturbation modes are characterized by distinct dispersion relations which describe the poles positions of the corresponding retarded Green functions [17].

Let us first examine the behavior of the shear mode for

fluids with zero chemical potential. The Euler identity reads  $\epsilon + P = Ts$ , where  $\epsilon$  is the energy density,  $P$  is the pressure,  $T$  is the temperature, and  $s$  is the entropy density of the fluid. The dispersion relation for a shear wave with frequency  $\omega$  and wave vector  $k \equiv 2\pi/\lambda$  is given by [26, 27]:

$$\omega(k)_{\text{shear}} = -i\frac{\eta}{Ts}k^2 + O\left(\frac{\eta^3 k^4}{s^3 T^3}\right), \quad (3)$$

where  $\eta$  is the shear viscosity coefficient of the standard first-order hydrodynamics. The correction term becomes small in the  $\eta/s \ll 1$  limit, the case of most interest here.

The imaginary part of the dispersion relation entails a damping of the perturbation mode. Its magnitude therefore quantifies the intrinsic ability of a fluid to dissipate perturbations and to approach thermal equilibrium.

It is important to realize that hydrodynamics is actually an effective theory. In the most common applications of hydrodynamics the underlying microscopic theory is a kinetic theory. In this case the microscopic scale which limits the validity of the effective hydrodynamic description is the mean free path  $l_{\text{mfp}}$  [26]. More generally, the underlying microscopic theory is a quantum field theory, which might not necessarily admit a kinetic description. In these cases, the role of the parameter  $l_{\text{mfp}}$  is played by some typical microscopic scale like the inverse temperature:  $l_{\text{mfp}} \sim T^{-1}$ . One therefore expects to find a breakdown of the effective hydrodynamic description at spatial and temporal scales of the order of [26]

$$l \sim \tau \sim T^{-1}. \quad (4)$$

Below we shall make this statement more accurate. It should be noted that the relation (4) may be modulated by a function of the dimensionless parameters of the theory (if any).

At this point, it is worth emphasizing that the conjectured KSS bound is based on holographic calculations of the shear viscosity for strongly coupled quantum field theories with gravity duals. These holographic arguments serve to connect quantum field theory with gravity. This fact indicates that a derivation of a KSS-like bound may require use of the still nonexistent theory of quantum gravity [10]. This may seem as bad news for our aspirations to prove (a refined version of) the KSS bound. But one need not lose heart—there is general agreement that black hole entropy reflects some aspect of the elusive theory of quantum gravity [10].

The realization that a black hole is endowed with well-defined entropy  $S_{\text{BH}} = A/4$ , where  $A$  is the surface area of the black hole [28, 29], has led to the formulation of the generalized second law (GSL) of thermodynamics. The GSL is a unique law of physics that bridges thermodynamics and gravity [10, 28, 29]. It asserts that in any interaction of a black hole with an ordinary matter, the sum of the entropies (matter+hole) never decreases. One

of the most remarkable predictions of the GSL is the existence of a universal entropy bound [30, 31]. According to this universal bound, the entropy contained in a given volume should be bounded from above:

$$S \leq 2\pi R E, \quad (5)$$

where  $R$  is the effective radius of the system and  $E$  is its total energy.

Furthermore, the generalized second law allows one to derive in a simple way two important new quantum bounds:

- The universal relaxation bound [32, 33, 34]. This bound asserts that the relaxation time of a perturbed thermodynamic system is bounded from below by

$$\tau \geq 1/\pi T, \quad (6)$$

where  $T$  is the temperature of the system. This bound can be regarded as a quantitative formulation of the third law of thermodynamics. One can also write this bound as  $\Im\varpi \leq 1/2$ , where  $\varpi \equiv \omega/2\pi T$ . The connection between the universal relaxation bound (6) and the Bekenstein entropy bound (5) is established in Ref. [32].

- A closely related conclusion is that thermodynamics can not be defined on arbitrarily small length scales. The minimal length scale (radius)  $\ell$  for which a consistent thermodynamic description is available is given by  $\ell_{\text{min}} = 1/2\pi T$ , see Refs. [10, 35, 36].

The longest wavelength which can fit into a space region of effective radius  $\ell$  is  $\lambda_{\text{max}} = 2\pi\ell$ . Thus, the GSL predicts that an effective hydrodynamic description is limited to perturbation modes with wavelengths larger than  $2\pi\ell_{\text{min}} = T^{-1}$ . This limit agrees with the one found from the heuristic argument presented above [26]. The breakdown of the effective hydrodynamic description for perturbation modes with wavenumbers  $k$  larger than  $2\pi T$  should manifest itself in the hydrodynamic dispersion relation (3) [see also Eq. (8) below]. This breakdown may reveal itself in two distinct ways:

- Short relaxation times which violate the universal relaxation bound (6), or
- Superluminal sound propagation which violates causality (this is not relevant for the shear mode).

A lower bound on the ratio  $\eta/s$  can be inferred by substituting  $q \equiv k/2\pi T = 1$  in the shear dispersion relation (3) and requiring that  $\Im\varpi \geq 1/2$  for this limiting value of the wavenumber. As discussed above, the GSL predicts that the effective hydrodynamic description breaks down for short wavelength perturbations with  $q > 1$ . This

should be reflected in the hydrodynamic shear dispersion relation in the form of a violation of the universal relaxation bound (6). Explicitly, these wavenumbers should be characterized by  $\Im\varpi > 1/2$ . This physical condition leads to the simple bound

$$\frac{\eta}{s} + O\left(\frac{\eta^3}{s^3}\right) \geq \frac{1}{4\pi}. \quad (7)$$

Let us now examine the sound perturbation mode. The sound dispersion relation for conformal field theories is given by [26, 27, 38]:

$$\Re\omega(k)_{\text{sound}} \simeq \pm v_s k \pm \frac{\Gamma}{v_s} \left( v_s^2 \tau_\pi - \frac{\Gamma}{2} \right) k^3, \quad (8)$$

where  $v_s = \sqrt{dP/d\epsilon} = 1/\sqrt{d}$  ( $d$  is the number of spatial dimensions),  $\Gamma = \frac{d-1}{d} \frac{\eta}{Ts}$ , and  $\tau_\pi$  is a relaxation coefficient whose origin is in the second-order causal hydrodynamics (see details below). The imaginary part of the sound dispersion relation is given by

$$\Im\omega(k)_{\text{sound}} = -i \frac{d-1}{d} \frac{\eta}{Ts} k^2 + \dots \quad (9)$$

A frequently used formalism for second-order hydrodynamics is the ‘Müller-Israel-Stewart’ formalism [39, 40, 41]. This extension of the first-order hydrodynamics attempts to repair problems that the first order theory has with causality, and necessarily introduces a set of new transport coefficients like  $\tau_\pi$ . This coefficient has the dimension of time, and it is often referred to as a relaxation time (although that is somewhat of a misnomer [42]). This new transport coefficient has been the focus of much recent attention. In particular, several groups have calculated this coefficient for various strongly coupled field theories [26, 27, 43, 44, 45].

The expected breakdown of the effective hydrodynamic description for sound mode perturbations with short wavelengths  $q > 1$  can manifest itself in two distinct ways: (1) A superluminal sound propagation with  $v_g = d\Re\varpi/dq > 1$ , or (2) Short relaxation times (characterized by  $\Im\varpi > 1/2$ ) which violate both the universal relaxation bound (6) and the GSL. Either one of these two options is by itself sufficient to infer a breakdown of the effective hydrodynamic description. Taking cognizance of the dispersion relations (8) and (9), and requiring that  $v_g \geq 1$  or  $\Im\varpi \geq 1/2$  for the limiting wavenumber  $q = 1$ , one obtains the lower bounds

$$\frac{\eta}{s} \left( \tau_\pi T - \frac{\eta}{s} \right) \geq \frac{\sqrt{3}-1}{8\pi^2} \quad \text{or} \quad \frac{\eta}{s} + O\left(\frac{\eta^3}{s^3}\right) \geq \frac{3}{8\pi}, \quad (10)$$

for the physically interesting case of field theories in three spatial dimensions.

It is instructive to check the validity of this new bound (10) against known results. For example, the canonical model of strongly coupled finite temperature  $\mathcal{N} = 4$  supersymmetric  $SU(N_c)$  Yang Mills theory in the limit

of large  $N_c$  is characterized by the well-known ratio  $\eta/s = 1/4\pi$ . Most recently, several groups have calculated the relaxation coefficient  $\tau_\pi$  of the second-order causal hydrodynamics and found  $\tau_\pi T = (2 - \ln 2)/2\pi$  for this model [26, 27, 43, 44, 45]. Substituting these two values into the l.h.s of (10), one may directly confirm the validity of the inequality. In fact, this canonical model is remarkably close ( $\sim 4\%$ ) of saturating the bound (10).

The new bound (10) combines the viscosity coefficient  $\eta$  of first-order hydrodynamics with the relaxation coefficient  $\tau_\pi$  of second-order hydrodynamics. From this bound one may also infer a concrete lower bound on the value of the relaxation coefficient  $\tau_\pi$  of the second-order causal hydrodynamics:

$$(\tau_\pi T)^2 \geq \frac{\sqrt{3}-1}{2\pi^2}. \quad (11)$$

This inequality should be satisfied by theories characterized by  $\eta/s < 3/8\pi$ .

In summary, we have given support to the idea that a lower bound on the viscosity to entropy ratio  $\eta/s$  may possibly be inferred from the generalized second law of thermodynamics. The bound (7) may be a bit more liberal than the originally conjectured KSS bound (1). Being a direct consequence of the generalized second law of thermodynamics, this bound is expected to be of general validity.

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